Sign changes of the error term of the Piltz divisor problem

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K-FOLD DIVISOR FUNCTIONS AND Δ_k

For an integer $k \geq 2$, define

$$d_k(n) = \sum_{n_1 \cdots n_k = n} 1$$

and

$$\Delta_k(x) = \sum_{n \leq x} d_k(n) - \text{Res}_{s=1}\left(\frac{\zeta^k(s)x^s}{s}\right).$$

The Piltz divisor problem is to determine the smallest α_k such that

$$\Delta_k(x) \ll x^{\alpha_k + \epsilon}$$

for all $\epsilon > 0$. Titchmarsh conjectured $\alpha_k = \frac{1}{2} - \frac{1}{2k}$.

TONG'S INTERMEDIATE VALUE THEOREM

Tong(1955) provided the existence of constants a_k and b_k such that $|y| \leq a_k X^{\frac{1}{2} - \frac{1}{2k}}$, then

$$\Delta_k(x) = y \text{ for some } x \in [X, X + b_k X^{1-\frac{1}{k}}].$$

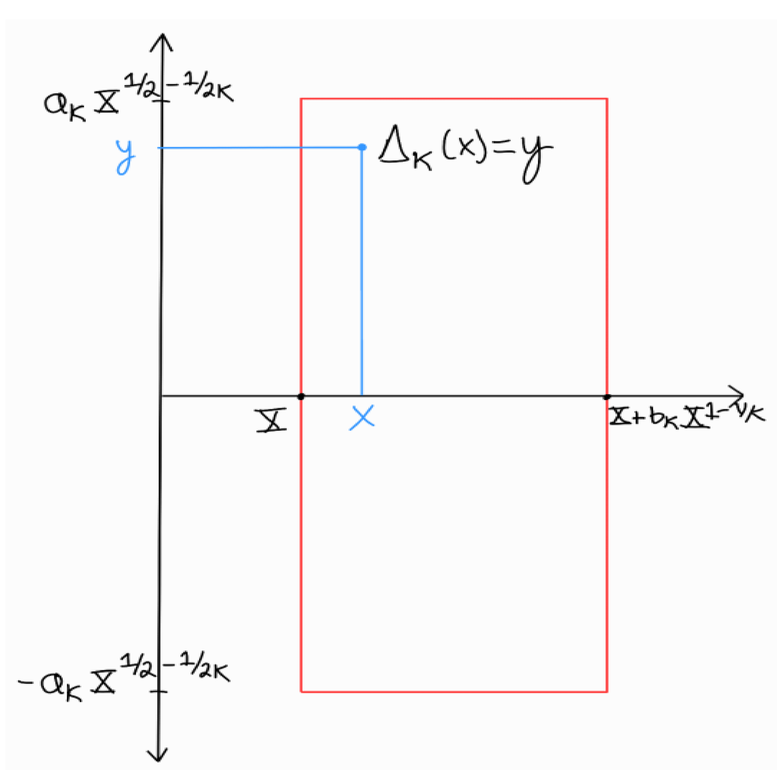

Figure 1. Tong's intermediate value theorem

Heath-Brown and Tsang(1994) showed that for some $c > 0$, there are at least $\gg \sqrt{X}(\log X)^5$ disjoint subintervals of [X,2X], each of length $c\sqrt{X}(\log X)^{-5}$, such that $|\Delta_2(x)| \gg x^{1/4}$ for all x in any of the subintervals. In particular, one can think of Heath-Brown's and Tsang's result saying in the case of $k = 2$ that Tong's is best possible up to some log factors.

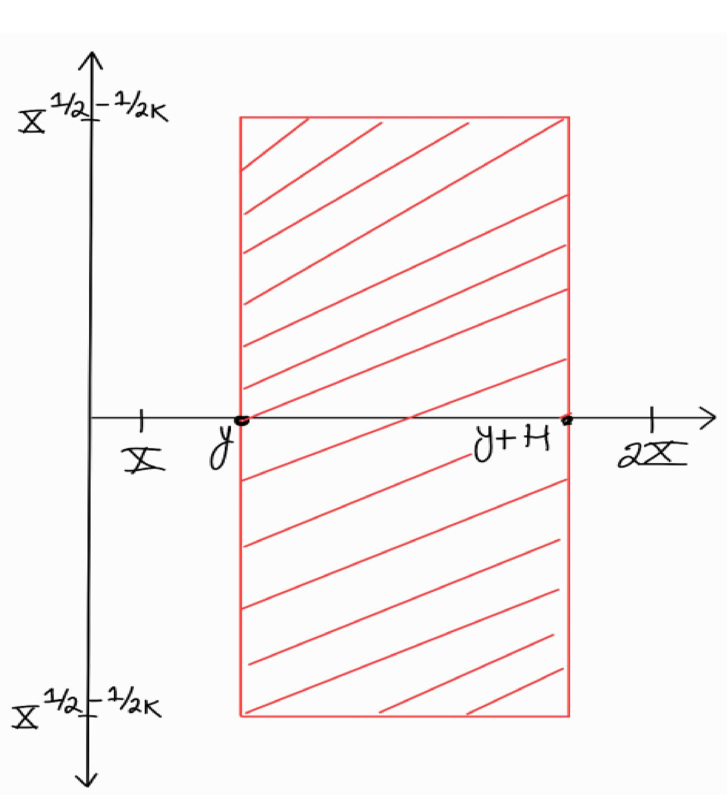

Figure 2. Intervals with no sign changes

RESULTS

Theorem (Baluyot and C.)

Assume the Lindelöf Hypothesis and let $k \geq 3$ be an integer. There are at least $\gg X^{\frac{1}{k(k-1)} - \varepsilon}$ disjoint subintervals of $[X, 2X]$, each of length $X^{1-\frac{1}{k}-\varepsilon}$, such that $|\Delta_k(x)| \gg x^{\frac{1}{2}-\frac{1}{2k}}$ for all x in any of the subintervals. In particular, $\Delta_k(x)$ does not change sign in any of the subintervals.

Theorem (Baluyot and C.)

Assume the Riemann Hypothesis and let $k \geq 3$ be an integer. There are at least $\gg X^{\frac{1}{k(k-1)} - \varepsilon}$ disjoint subintervals of $[X, 2X]$, each of length $X^{1-\frac{1}{k}}(\log X)^{-k^2-2}$, such that $|\Delta_k(x)| \gg x^{\frac{1}{2}-\frac{1}{2k}}$ for all x in any of the subintervals. In particular, $\Delta_k(x)$ does not change sign in any of the subintervals.

Who(year)	Assumption	k	H	Sign change
Tong(1955)	None	≥ 2	$X^{1-\frac{1}{k}}$	Yes
Heath-Brown and Tsang(1994)	None	$= 2$	$\sqrt{X}(\log X)^{-5}$	No
Cao, Tanigawa and Zhai(2016)	None	$= 3$	$X^{\frac{1}{2}-\varepsilon}$	No
CTZ(2016)	LH	$= 3$	$X^{\frac{2}{3}-\varepsilon}$	No
Baluyot and C.(2023)	LH	≥ 3	$X^{1-\frac{1}{k}-\varepsilon}$	No
BC(2023)	RH	≥ 3	$X^{1-\frac{1}{k}}(\log X)^{-k^2-2}$	No

DETECTION METHOD OF HEATH-BROWN AND TSANG

To detect intervals without sign changes we will consider the following set:

$$S := \left\{x \in [X, 2X] : |\Delta_k(x)|^2 > \sup_{0 \leq h \leq H} |\Delta_k(x+h) - \Delta_k(x)|^2\right\}.$$

Note that if $x \in S$, then $\Delta_k(x+h)$ has the same sign as $\Delta_k(x)$ for all $h \leq H$. By the definition of S and Cauchy-Schwarz,

$$\int_X^{2X} |\Delta_k(x)|^2\, dx - \int_X^{2X} \sup_{0 \leq h \leq H} |\Delta_k(x+h) - \Delta_k(x)|^2\, dx$$
$$\leq \int_S \left(|\Delta_k(x)|^2 - \sup_{0 \leq h \leq H} |\Delta_k(x+h) - \Delta_k(x)|^2\right) dx$$
$$\leq \int_S |\Delta_k(x)|^2\, dx$$
$$\leq \text{meas}(S)^{1/2}\left(\int_X^{2X} |\Delta_k(x)|^4\, dx\right)^{1/2}.$$

Hence, to get a lower bound for meas(S), we need a lower bound for the second moment Δ_k and upper bounds for the fourth moment of Δ_k and the variance of sums of $d_k(n)$ in short intervals.

MEAN SQUARE DIFFERENCE OF Δ_k

Theorem(Baluyot and C.)

Assume the Lindelöf Hypothesis. If $k \geq 3$ is an integer and $1 \leq h \leq X$, then

$$\int_X^{2X} \Big(\Delta_k(x+h) - \Delta_k(x)\Big)^2\, dx \ll X^{1+\varepsilon} h.$$

Theorem(Baluyot and C.)

Assume the Riemann Hypothesis. If $k \geq 3$ is an integer and $1 \leq h \leq X$, then

$$\int_X^{2X} \Big(\Delta_k(x+h) - \Delta_k(x)\Big)^2\, dx \ll X h \log^{k^2}\left(\frac{X}{h}\right).$$

Outline of proof:

- View Δ_k as a Fourier Transform:

$$\Delta_k(x) = \sum_{n \leq x} d_k(n) - \text{Res}_{s=1}\left(\frac{\zeta^k(s)x^s}{s}\right) = \lim_{Y \to \infty} \frac{1}{2\pi i} \int_{\frac{1}{2}-iY}^{\frac{1}{2}+iY} \frac{x^s}{s} \zeta^k(s)\, ds.$$

- Apply Plancherel's theorem

$$\int_0^\infty \left|\Delta_k\left(x + \frac{x}{T}\right) - \Delta_k(x)\right|^2 \frac{dx}{x^2} = \frac{1}{\pi} \int_0^\infty \left|\left(\frac{(1+\frac{1}{T})^{(\frac{1}{2}+it)} - 1}{\frac{1}{2}+it}\right) \zeta^k(\tfrac{1}{2}+it)\right|^2 dt \ll \frac{1}{T^{1-\varepsilon}}.$$

- Truncate the integral to see

$$\int_X^{2X} \left|\Delta_k\left(x + \frac{x}{T}\right) - \Delta_k(x)\right|^2 dx \ll \frac{X^2}{T^{1-\varepsilon}}.$$

- Apply a lemma due to Saffari and Vaughn(1977):

$$\int_{X/2}^X |\Delta_k(x+h) - \Delta_k(x)|^2\, dx \leq \frac{2X}{h} \int_0^{8h/X} \int_0^X |\Delta_k(x + \beta x) - \Delta_k(x)|^2\, dx\, d\beta.$$

THE FOURTH MOMENT OF Δ_k

Theorem(Baluyot and C.)

Assume the Lindelöf Hypothesis. If $k \geq 3$ is an integer, then

$$\int_X^{2X} \big(\Delta_k(x)\big)^4\, dx \ll X^{3 - \frac{1}{k-1} + \varepsilon}.$$

Outline of proof:

We apply a formulation of Δ_k due to Lester(2016). Let $0 \leq \delta < 1/2$ be fixed. If $x, T \geq 1$ and $1 \leq Y \leq \min\{x, T\}$,

$$\Delta_k(x) = \frac{x^{\frac{1}{2}-\frac{1}{2k}}}{\pi\sqrt{k}} \sum_{n \leq \frac{1}{x}\left(\frac{Y}{2\pi}\right)^k} \frac{d_k(n)}{n^{\frac{1}{2}+\frac{1}{2k}}} \cos\left(2\pi k(nx)^{1/k} + \frac{(k-3)\pi}{4}\right) + \text{Re}\left\{\frac{1}{\pi i} \int_{\frac{1}{2}-\delta+iY}^{\frac{1}{2}-\delta+iT} \zeta^k(s) \frac{x^s}{s}\, ds\right\}$$
$$+ E = Q + I + E.$$

- Apply Tsang's(1991) method of using Voronoi summation + Erdös-Turán inequality to bound the third and fourth moment of Δ_2.
- Apply Lester's(2016) method together with the Riesz-Thorin Interpolation Theorem for the I term.