Sign changes of the error term in the Piltz divisor problem

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The Piltz divisor problem

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- $\sum_{n \leq x} d_k(n) = \operatorname*{Res}_{s=1} \left(\frac{\zeta^k(s) x^s}{s} \right) + \Delta_k(x) = x P_k(\log x) + \Delta_k(x)$.

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for some polynomial P_k of degree $k-1$.
- The Piltz divisor problem is to determine the smallest α_k such that

$$\Delta_k(x) \ll x^{\alpha_k + \varepsilon}$$

for all $\varepsilon > 0$.
Titchmarsh conjectured $\alpha_k = \gamma_k := \frac{1}{2} - \frac{1}{2k}$.

Progress on Dirichlet/Piltz Divisor Problem

Who(Year)	Result
Dirichlet(1849)	$\alpha_2 \leq \frac{1}{2}$
Huxley(2003)	$\alpha_2 \leq \frac{131}{416} \approx .3149$
Kolesnik(1981)	$\alpha_3 \leq \frac{43}{96} \approx .4479$
Ivić(1980's)	$\alpha_k \leq \frac{3}{4} - \frac{1}{k},$ for $3 \leq k \leq 8$.
Richert(1960)	$\alpha_k \leq 1 - ck^{-2/3}$
Ford(2002)	$\alpha_k \leq 1 - \frac{1}{3}\left(\frac{2}{(4.45)}\right)^{\frac{2}{3}} \leq 1 - .196k^{-\frac{2}{3}},$ for large k.
Bellotti and Yang(2023)	$\alpha_k \leq 1 - 1.889k^{-\frac{2}{3}},$ for large k.

Size and fluctuations of $\Delta_k(x)$

- Soundararajan (2003), building on ideas of Hafner, has shown

$$\Delta_k(x) = \Omega\left((x \log x)^{\frac{1}{2}-\frac{1}{2k}} (\log\log x)^{\frac{k+1}{2k}(k^{2k/(k+1)}-1)} (\log\log\log x)^{-\frac{1}{2}-\frac{k-1}{4k}}\right).$$

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- Tong (1955) proved the existence of constants a_k and b_k such that if $|y| \le a_k X^{\frac{1}{2}-\frac{1}{2k}}$, then

$$\Delta_k(x) = y \quad \text{for some} \quad x \in \big[X, X + b_k X^{1-\frac{1}{k}}\big].$$

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$$\Delta_k(x) = y \quad \text{for some} \quad x \in \left[X, X + b_k X^{1-\frac{1}{k}}\right].$$

- In particular, $\Delta_k(x)$ changes sign at least once in the interval $[X, X + b_k X^{1-\frac{1}{k}}]$. Question: Is this best possible?

Intervals with no sign changes of $\Delta_k(x)$

Let X be a large parameter.

• Tong (1955): $\Delta_k(x)$ changes sign in $[X, X + b_k X^{1-\frac{1}{k}}]$

• Heath-Brown and Tsang (1994): For some constant $c > 0$, there are at least $\gg \sqrt{X}(\log X)^5$ disjoint subintervals of $[X, 2X]$, each of length $c\sqrt{X}(\log X)^{-5}$, such that $|\Delta_2(x)| \gg x^{1/4}$ for all x in any of the subintervals. In particular, $\Delta_2(x)$ does not change sign in any of the subintervals.

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- Cao, Tanigawa, and Zhai(2016): For some constant $c > 0$, there are at least $\gg X^{1/2-\varepsilon}$ disjoint subintervals of $[X, 2X]$, each of length $X^{1/2-\varepsilon}$, such that $|\Delta_3(x)| \geq cx^{1/3}$ for all x in any of the subintervals. In particular, $\Delta_3(x)$ does not change sign in any of the subintervals. Assuming Lindelöf, there are at least $\gg X^{1/3-\varepsilon}$ of disjoint intervals of length $X^{2/3-\varepsilon}$.

Intervals with no sign changes of $\Delta_k(x)$

Theorem (Baluyot and C.)

Assume the Lindelöf Hypothesis and let $k \geq 3$ be an integer. There are at least $\gg X^{\frac{1}{k(k-1)}-\varepsilon}$ disjoint subintervals of $[X, 2X]$, each of length $X^{1-\frac{1}{k}-\varepsilon}$, such that $|\Delta_k(x)| \gg x^{\frac{1}{2}-\frac{1}{2k}}$ for all x in any of the subintervals. In particular, $\Delta_k(x)$ does not change sign in any of the subintervals.

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Remark

For $k = 3$ in the first theorem we recover the length of disjoint subintervals proved by Cao, Tanigawa, and Zhai(2016); however, we do not recover the lower bound on the number of subintervals.

Intervals with no sign changes

Who(year)	Assumption	k	H	Sign change
Tong(1955)	None	≥ 2	$X^{1-\frac{1}{k}}$	Yes
Heath-Brown and Tsang(1994)	None	$= 2$	$\sqrt{X}(\log X)^{-5}$	No
Cao, Tanigawa and Zhai(2016)	None	$= 3$	$X^{\frac{1}{2}-\varepsilon}$	No
CTZ(2016)	LH	$= 3$	$X^{\frac{2}{3}-\varepsilon}$	No
Baluyot and C.(2023)	LH	≥ 3	$X^{1-\frac{1}{k}-\varepsilon}$	No
BC(2023)	RH	≥ 3	$X^{1-\frac{1}{k}}(\log X)^{-k^2-2}$	No

The detection method of Heath-Brown and Tsang

Let $S := \left\{ x \in [X, 2X] : \ |\Delta_k(x)|^2 > \sup_{0 \le h \le H} |\Delta_k(x+h) - \Delta_k(x)|^2 \right\}$ If $x \in S$, then $\Delta_k(x+h)$ has the same sign as $\Delta_k(x)$ for all $h \le H$. We use the definition of S and Cauchy-Schwarz to deduce that

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$$\int_X^{2X} |\Delta_k(x)|^2 \, dx - \int_X^{2X} \sup_{0 \le h \le H} |\Delta_k(x+h) - \Delta_k(x)|^2 \, dx$$
$$\le \int_S \left(|\Delta_k(x)|^2 - \sup_{0 \le h \le H} |\Delta_k(x+h) - \Delta_k(x)|^2 \right) dx$$
$$\le \int_S |\Delta_k(x)|^2 \, dx$$
$$\le \operatorname{meas}(S)^{1/2} \left(\int_X^{2X} |\Delta_k(x)|^4 \, dx \right)^{1/2}$$

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$$\le \int_S \left(|\Delta_k(x)|^2 - \sup_{0 \le h \le H} |\Delta_k(x+h) - \Delta_k(x)|^2 \right) dx$$

$$\le \int_S |\Delta_k(x)|^2 \, dx$$

$$\le \text{meas}(S)^{1/2} \left(\int_X^{2X} |\Delta_k(x)|^4 \, dx \right)^{1/2}$$

To get a lower bound for meas(S), we need a lower bound for the second moment of Δ_k and upper bounds for the fourth moment of Δ_k and the variance of sums of $d_k(n)$ in short intervals.

The second moment of Δ_k

- Cramér (1922): $\int_0^X \left(\Delta_2(x)\right)^2 dx \sim A_2 X^{3/2}$ for some constant A_2.

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- Cramér (1922): $\int_0^X (\Delta_2(x))^2\, dx \sim A_2 X^{3/2}$ for some constant A_2.
- Tong (1956): If $k \geq 3$ is an integer, then for some constant A_k,

$$\int_0^X (\Delta_k(x))^2\, dx = A_k X^{2-\frac{1}{k}} + O\left(X^{2-\frac{3-4\sigma_k}{2k(1-\sigma_k)-1}+\varepsilon}\right),$$

where $\sigma_k \geq \frac{1}{2}$ satisfies

$$\int_0^T |\zeta(\sigma_k + it)|^{2k}\, dt \ll T^{1+\varepsilon}.$$

In particular, the Lindelöf Hypothesis implies an asymptotic formula.

The variance of sums of $d_2(n)$ in short intervals

- Jutila (1984): If $X^{\varepsilon} \ll h \leq \frac{1}{2}\sqrt{X}$, then

$$\int_X^{2X} \Big(\Delta_2(x+h) - \Delta_2(x)\Big)^2 \, dx \ll Xh \log^3 \left(\frac{\sqrt{X}}{h}\right).$$

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- Ivić (2009): If $1 \ll h \le \frac{1}{2}\sqrt{X}$, then for some constants $c_0, \ldots, c_3$,

$$\int_X^{2X} \Big(\Delta_2(x+h) - \Delta_2(x)\Big)^2 dx = Xh \sum_{j=0}^{3} c_j \log^j \left(\frac{\sqrt{X}}{h}\right) + O\Big(X^{\frac{1}{2}+\varepsilon} h^2 + X^{1+\varepsilon} h^{1/2}\Big).$$

Note that this is an asymptotic formula for $X^\varepsilon \ll h \ll X^{\frac{1}{2}-\varepsilon}$.

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Let $k \geq 3$, and let $\sigma_k \geq \frac{1}{2}$ satisfy $\int_0^T |\zeta(\sigma_k + it)|^{2k}\, dt \ll T^{1+\varepsilon}$.

• Ivić (2009): If $\frac{1}{2} < \sigma_k < 1$, $2\sigma_k - 1 < \theta < 1$, and $X^\theta \leq h \ll X^{1-\varepsilon}$, then

$$\int_X^{2X} \Big(\Delta_k(x+h) - \Delta_k(x)\Big)^2 dx \ll X^{1-\frac{1}{3}(\theta - 2\sigma_k + 1)+\varepsilon} h^2.$$

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If $\sigma_k = \frac{1}{2}$ and $X^\varepsilon \ll h \ll X^{1-\varepsilon}$, then

$$\int_X^{2X} \Big(\Delta_k(x+h) - \Delta_k(x)\Big)^2 dx \ll X^{1+\varepsilon} h^{4/3}.$$

Cao,Tanigawa, and Zhai(2016)

- Unconditionally, for $h \leq X$,

$$\int_X^{2X} (\Delta_3(x+h) - \Delta_3(x))^2 dx \ll X^{\varepsilon}(Xh + X^{4/3}h^{1/3} + X^{14/9})$$

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- Assume LH. For $h \leq X$

$$\int_X^{2X} (\Delta_k(x+h) - \Delta_k(x))^2 dx \ll X^{\varepsilon}(Xh + X^{2-3/k}).$$

Lester (2016)

- Unconditionally, if $2 \leq L \ll X^{\frac{1}{12}-\varepsilon}$, then for some constant C_3,

$$\int_X^{2X} \left(\Delta_3\left(x + \frac{x^{2/3}}{L}\right) - \Delta_3(x)\right)^2 dx = C_3 \frac{X^{5/3}}{L} (\log L)^8 \left(1 + O\left(\frac{1}{\log L}\right)\right).$$

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- Assume LH. If $k \geq 3$ and $2 \leq L \ll X^{\frac{1}{k(k-1)}-\varepsilon}$, then for some constant C_k,

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- Essentially, when $X^{1-\frac{1}{k-1}+\varepsilon} \ll h \leq \frac{1}{2} X^{1-\frac{1}{k}}$

$$\int_X^{2X} \left(\Delta_k(x+h) - \Delta_k(x) \right)^2 dx \sim CXh(\log X)^{k^2-1}$$

.

The variance of sums of $d_k(n)$ in short intervals

- Conjecture (Keating, Rodgers, Roditty-Gershon, and Rudnick, 2018): If $h = X^\delta$ with $0 < \delta < 1 - \frac{1}{k}$ fixed, then

$$\int_X^{2X} \Big(\Delta_k(x+h) - \Delta_k(x)\Big)^2 \, dx \sim \mathcal{P}_k(\delta) X h (\log X)^{k^2-1},$$

where $\mathcal{P}_k(\delta)$ is a (specific) piecewise polynomial function of δ of degree $k^2 - 1$.

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• By examining the function field case, KRRR have found an interesting connection between this variance and averages of coefficients of characteristic polynomials of random matrices.

• This agrees with Lester's theorem for $1 - \frac{1}{k-1} < \delta < 1 - \frac{1}{k}$.

Mean square of Δ_k under LH

Theorem (Baluyot and C.)

Assume the Lindelöf Hypothesis. If $k \geq 3$ is an integer and $0 \leq h \leq X$, then

$$\int_X^{2X} \Big(\Delta_k(x+h) - \Delta_k(x)\Big)^2 dx \ll X^{1+\varepsilon} h.$$

Remark

Our result recovers Cao, Tanigawa, Zhai(2016) in the case of $k = 3$, and improves on it for higher k.

- For $h \leq X$

$$\int_X^{2X} (\Delta_k(x+h) - \Delta_k(x))^2 dx \ll X^{\varepsilon}(Xh + X^{2-3/k}).$$

Mean square of Δ_k under RH

We can use bounds for the moments of zeta due to Harper(2013).

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Remark

This bound is one factor of $\log X$ larger than the conjectured order.

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- Assume the LH. We use a method developed by Selberg in studying primes in short intervals. Then, we apply a lemma of Saffari and Vaughan.

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• Apply Perron and then move the line of integration to $\mathrm{Re}(s) = \frac{1}{2}$:

$$\Delta_k(x) = \sum_{n \le x} d_k(n) - \operatorname*{Res}_{s=1} \left(\frac{\zeta^k(s) x^s}{s} \right) = \lim_{Y \to \infty} \frac{1}{2\pi i} \int_{\frac{1}{2}-iY}^{\frac{1}{2}+iY} \frac{x^s}{s} \zeta^k(s) \, ds$$

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We may thus view $\Delta_k(x)$ as a Fourier transform.

• Apply Plancherel's theorem, make a change of variables:

$$\int_0^\infty \left| \Delta_k \left(x + \frac{x}{T} \right) - \Delta_k(x) \right|^2 \frac{dx}{x^2}$$
$$= \frac{1}{\pi} \int_0^\infty \left| \left(\frac{(1 + \frac{1}{T})^{(\frac{1}{2} + it)} - 1}{\frac{1}{2} + it} \right) \zeta^k(\tfrac{1}{2} + it) \right|^2 dt \ll \frac{1}{T^{1-\varepsilon}}.$$

A lemma of Saffari and Vaughan

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- Formulation due to Goldston and Suriajaya:

Lemma (Saffari and Vaughan(1977))

For any integrable function f, if $0 \leq h \leq X/4$, then

$$\int_{X/2}^{X} |f(x+h) - f(x)|^2 \, dx \leq \frac{2X}{h} \int_0^{8h/X} \int_0^X |f(x+\beta x) - f(x)|^2 \, dx \, d\beta.$$

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$$\int_0^X \left(\Delta_3(x)\right)^4 dx \ll X^{\frac{235}{96}+\varepsilon}.$$

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- Ivić (1985):

$$\int_0^X \big(\Delta_3(x)\big)^4 \, dx \ll X^{\frac{235}{96}+\varepsilon}.$$

- Ivić and Zhai(2010):

$$\int_X^{2X} (\Delta_k)^4(x)dx \ll X^{\varepsilon}(X^{3-1/k} + X^{(10k-11)/(2k+1)}).$$

The fourth moment of Δ_k

Theorem (Baluyot and C.)

Assume the Lindelöf Hypothesis. If $k \geq 3$ is an integer, then

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Remark

- We expect the true order to be $X^{3-\frac{2}{k}}$ times some power of $\log X$.
- The $k = 3$ case is unconditional, but weaker than the theorem of Ivić (1985).

The detection method of Heath-Brown and Tsang

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$$X^{2-1/k} \ll \int_X^{2X} |\Delta_k(x)|^2 \, dx - \int_X^{2X} \sup_{0 \le h \le H} |\Delta_k(x+h) - \Delta_k(x)|^2 \, dx$$

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This gives there are at least $\gg X^{\frac{1}{k(k-1)} - \varepsilon}$ disjoint subintervals of $[X, 2X]$ of length $X^{1-1/k-\varepsilon}$ with no sign changes.

Remark

The $2 - 1/k$ in the lower bound for the measure of S is why Ivić and Zhai(2010) is insufficient for the theorem.

Thank you!